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A graph associated to a lattice

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Abstract In this paper, we associate a simple graph to a lattice $\mathcal{L}$, in which the vertex set is being the set of all elements of $\mathcal{L}$, and two distinct vertices x and y are adjacent if $x \vee y \in S$, when S is a multiplicatively closed subset of $\mathcal{L}$. We denote this graph by $\Gamma_S(\mathcal{L})$. We study some properties of $\Gamma_S(\mathcal{L})$. Moreover, we investigate the planarity of $\Gamma_S(\mathcal{L})$, whenever S is a saturated multiplicatively closed subset of $\mathcal{L}$.

Keywords Lattice · Planar graph

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1 Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. There are many papers on assigning a graph to a ring, see for example [2,3,6,4]. One of these graphs is the total graph. Let R be a commutative ring. The total graph of R is a simple graph with vertex set R, and two distinct vertices a and b are adjacent if $a + b$ is a zero divisor of R (see for

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example [1,2]). By using this idea, for a multiplicatively closed subset S of R, in [5], the authors defined a graph on R, denoted by $\Gamma_S(R)$, with vertices as elements of R, and two distinct vertices a and b are adjacent if and only if $a+b \in S$. In this paper, we generalize the concept of $\Gamma_S(R)$ for a lattice. Recall that a *lattice* is an algebra $\mathcal{L} = (L, \vee, \wedge)$ satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a = a, a \vee a = a$,
2. $a \wedge b = b \wedge a, a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c), (a \vee b) \vee c = a \vee (b \vee c)$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

There is an equivalent definition for a lattice (see for example [8, Theorem 2.1]). To do this, for a lattice $\mathcal{L}$, one can define an order $\leqslant$ on $\mathcal{L}$ as follows: For any $a, b \in \mathcal{L}$, we set $a \leq b$ if and only if $a \wedge b = a$. Then $(\mathcal{L}, \leq)$ is an ordered set in which every pairs of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, g.l.b.$(a, b) \in P$ and l.u.b.$(a, b) \in P$. For each a and b in P, we define $a \wedge b :=$ g.l.b.(a, b) and $a \vee b :=$ l.u.b.(a, b). Clearly $(P, \wedge, \vee)$ is a lattice.

The lattice $\mathcal{L}$ is said to be *bounded* if there are elements 0 and 1 such that

$$0 \wedge x = 0 \quad \text{and} \quad x \vee 1 = 1$$

for all $x \in \mathcal{L}$. Let S be a nonempty subset of $\mathcal{L}$. We say that S is a multiplicatively closed subset of $\mathcal{L}$ if $x \wedge y \in S$, for all x and y of S. Also, we say that a subset S of $\mathcal{L}$ is saturated if $x \wedge y \in S$ if and only if $x, y \in S$.

For a multiplicatively closed subset of $\mathcal{L}$, we define $\Gamma_S(\mathcal{L})$ as a simple graph, with vertex-set $\mathcal{L}$ and two distinct vertices x and y being adjacent if and only if $x \vee y \in S$.

In the second section, we study some basic properties of the graph $\Gamma_S(\mathcal{L})$ such as connectivity, diameter and completeness whenever $\mathcal{L}$ is a bounded lattice. In the third section, for bounded lattices, we investigate $\Gamma_S(\mathcal{L})$, whenever S is a saturated multiplicatively closed subset of $\mathcal{L}$ and, in the final section, we study the planarity of $\Gamma_S(\mathcal{L})$ where $\mathcal{L}$ is a bounded lattice.

Now, we start to remind a belief necessary background of lattice theory from [8]. Let x and y be two distinct elements of $\mathcal{L}$. Whenever $x \leqslant y$ and there is no element z in $\mathcal{L}$ such that $x \leqslant z \leqslant y$, we say y *covers* x. An element $a \in \mathcal{L}$ is an *atom* of lattice $\mathcal{L}$ if it covers 0. Also, an element $m \in \mathcal{L}$ is a *coatom* of lattice $\mathcal{L}$ if 1 covers it. We denote the set of all coatoms of $\mathcal{L}$ by Coatom $(\mathcal{L})$ and the set of atoms of $\mathcal{L}$ by Atom $(\mathcal{L})$. An ideal I of $\mathcal{L}$ is a non-empty subset of $\mathcal{L}$ such that

(i) for all a and b of I, $a \vee b \in I$, and
(ii) for any $a \in I$ and $b \in \mathcal{L}$, $a \wedge b \in I$.

An ideal I of $\mathcal{L}$ is maximal if I is proper $(I \neq \mathcal{L})$ and the only ideal having I, as a proper subset, is $\mathcal{L}$. We use the notation $\mathrm{J}(\mathcal{L})$ for the jacobson radical of $\mathcal{L}$ which is the intersection of all maximal ideals of $\mathcal{L}$. Given a lattice $\mathcal{L}$ and $A \subseteq \mathcal{L}$. An element $x \in \mathcal{L}$ is a lower bound of A if $x \leq a$ for all $a \in A$. An upper bound is defined in a dual mannar. The set of all lower bounds of A is denoted by A^ℓ and the set of all upper bounds by A^u, where

$$A^{\ell} := \{x \in \mathcal{L};\ x \leq a \ \text{for all}\ a \in A\}$$

and

$$A^{u} := \{x \in \mathcal{L};\ a \leq x \ \text{for all}\ a \in A\}\,.$$

Now, we recall some definitions of graph theory from [7] which are needed in this paper. The *degree* of a vertex v in the graph X is the number of edges of X incident with v and denoted by $\deg(v)$. In a graph X with vertex-set $V(X)$, the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a, b) := \infty$. The *diameter* of a graph X is $\mathrm{diam}(X) = \sup\{\mathrm{d}(a, b) : a$ and b are distinct vertices of $X\}$. For two distinct vertices a and b in X, the notation $a - b$ means that a and b are adjacent. Also the *girth* of a graph X, denoted by $\mathrm{gr}(X)$, is the length of a shortest cycle in X if X has a cycle; otherwise, we set $\mathrm{gr}(X) := \infty$. A graph X is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. Also, we say that X is *totally disconnected* if no two vertices of X are adjacent. A *clique* of a graph is a complete subgraph of X and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$.

2 Basic properties of $\Gamma_S(\mathcal{L})$

Form now on $\mathcal{L}$ is a bounded lattice. In this section S is a multiplicatively closed subset of $\mathcal{L}$. We begin with the following lemma.

Lemma 2.1 *If* $0 \notin S$, *then* $|S \cap \mathrm{Atom}(\mathcal{L})| \leq 1$.

Proof Assume in contrary that $|S \cap \mathrm{Atom}(\mathcal{L})| \geq 2$. Let a_1 and a_2 be two distinct elements in $S \cap \mathrm{Atom}(\mathcal{L})$. Since S is a multiplicatively closed subset of $\mathcal{L}$, we have $a_1 \wedge a_2 \in S$ which implies that $0 \in S$. This is a required contradiction. Hence $|S \cap \mathrm{Atom}(\mathcal{L})| \leq 1$. □

Lemma 2.2 *The following statements hold.*

(i) If $0 \in S$, *then* $\deg(0) = |S| - 1$. *Otherwise,* $\deg(0) = |S|$.
(ii) If $1 \in S$, *then* $\deg(1) = |\mathcal{L}| - 1$. *Otherwise,* $\deg(1) = 0$.

Proof Since $0 \vee x = x$ and $1 \vee x = 1$, for all $x \in \mathcal{L}$, the statements (i) and (ii) hold. □

In the following theorem, we study the connectivity of $\Gamma_S(\mathcal{L})$.

Theorem 2.3 $\Gamma_S(\mathcal{L})$ *is connected if and only if* $1 \in S$. *Moreover, if* $\Gamma_S(\mathcal{L})$ *is connected, then* $\mathrm{diam}(\Gamma_S(\mathcal{L})) \leq 2$.

Proof Suppose that $1 \in S$. By part (ii) of Lemma 2.2, $\deg(1) = |\mathcal{L}| - 1$ which implies that $\Gamma_S(\mathcal{L})$ is connected.

Conversely, if $\Gamma_S(\mathcal{L})$ is connected, then $\deg(1) \neq 0$. Now, by part (ii) of Lemma 2.2, we have $1 \in S$.

The last statement is clear. □

Proposition 2.4 $\Gamma_S(\mathcal{L})$ *is complete if and only if* $S = \mathcal{L}$ *or* $S = \mathcal{L} \setminus \{0\}$.

Proof Assume that $\Gamma_S(\mathcal{L})$ is complete. So $\deg(0) = |\mathcal{L}| - 1$. By part (i) of Lemma 2.2, we have $|S| = |\mathcal{L}| - 1$ or $|S| - 1 = |\mathcal{L}| - 1$. Hence $S = \mathcal{L}$ or $S = \mathcal{L} \setminus \{x\}$, for some $x \in \mathcal{L}$. We claim that $x = 0$. Assume in contrary that $x \neq 0$. Since $0 \vee x = x$, the vertex 0 is not adjacent to x, which is impossible. So $x = 0$, and thus $S = \mathcal{L}$ or $S = \mathcal{L} \setminus \{0\}$.

The converse statement follows easily. □

Theorem 2.5 *The following statements hold.*

(i) The graph $\Gamma_S(\mathcal{L})$ *is regular if and only if it is either complete or totally disconnected.*
(ii) $\Gamma_S(\mathcal{L})$ *is a star graph if and only if* $S = \{1\}$ *or* $S = \{1, 0\}$ *and* $|\mathrm{Coatom}(\mathcal{L})| = 1$.

Proof (i) Suppose that $\Gamma_S(\mathcal{L})$ is regular and that is not totally disconnected. By part (ii) of Lemma 2.2, we have $1 \in S$. Then $\deg(1) = |\mathcal{L}| - 1$. Since $\Gamma_S(\mathcal{L})$ is regular, $\deg(x) = |\mathcal{L}| - 1$, for all $x \in \mathcal{L}$. Hence $\Gamma_S(\mathcal{L})$ is complete.

The converse statement is clear.

(ii) Suppose that $\Gamma_S(\mathcal{L})$ is a star graph. Since $\Gamma_S(\mathcal{L})$ is connected, we have $1 \in S$, and so $\deg(1) = |\mathcal{L}| - 1$. Hence 1 is the center of $\Gamma_S(\mathcal{L})$. On the other hand, since 0 is adjacent to all element of S, we have $S = \{1\}$ or $S = \{1, 0\}$. Assume in contrary that $|\mathrm{Coatom}(\mathcal{L})| \geq 2$. Thus there exists distinct vertices m and m' in $\mathrm{Coatom}(\mathcal{L})$ which are adjacent in $\Gamma_S(\mathcal{L})$. This is impossible. Hence $|\mathrm{Coatom}(\mathcal{L})| = 1$.

The converse statement is clear. □

Proposition 2.6 *Assume that* $1 \in S$. *Then* $\mathrm{gr}(\Gamma_S(\mathcal{L})) \in \{3, \infty\}$.

Proof Suppose that $|\mathrm{Coatom}(\mathcal{L})| \geq 2$ and consider the cycle $1 - m - m' - 1$ in $\Gamma_S(\mathcal{L})$, where $m, m' \in \mathrm{Coatom}(\mathcal{L})$, to deduce that $\mathrm{gr}(\Gamma_S(\mathcal{L})) = 3$. Otherwise, $|\mathrm{Coatom}(\mathcal{L})| = 1$. Now we have the following cases:

(i) Suppose that $|S| = 1$. Then $S = \{1\}$ and so, by Proposition 2.5, the graph $\Gamma_S(\mathcal{L})$ is a star graph. Thus $\mathrm{gr}(\Gamma_S(\mathcal{L})) = \infty$.
(ii) Suppose that $|S| = 2$. If $S = \{0, 1\}$, then, by Proposition 2.5, $\Gamma_S(\mathcal{L})$ is a star graph and so $\mathrm{gr}(\Gamma_S(\mathcal{L})) = \infty$. Otherwise, $S = \{1, s\}$ where $s \neq 0$. Now, the cycle $1 - s - 0 - 1$ is the shortest cycle in the graph $\Gamma_S(\mathcal{L})$ which implies that $\mathrm{gr}(\Gamma_S(\mathcal{L})) = 3$.
(iii) Suppose that $|S| \geq 2$. Thus there is $s \in S$ such that $s \neq 0, 1$. Now, the cycle $1 - s - 0 - 1$ is the shortest cycle in the graph $\Gamma_S(\mathcal{L})$. So $\mathrm{gr}(\Gamma_S(\mathcal{L})) = 3$.

□

Let $\mathcal{A}$ be a chain in S. Since $x \vee y \in S$ for all $x, y \in \mathcal{A}$, we have the following proposition.

Proposition 2.7 $\omega(\Gamma_S(\mathcal{L})) \geq \max\{|\mathcal{A}|; \mathcal{A}\ \text{is a chain in}\ S\}$.

3 Basic properties of $\Gamma_S(\mathcal{L})$ where S is a saturated subset of the lattice $\mathcal{L}$

Throughout this section, S is a saturated subset of the lattice $\mathcal{L}$. It is easy to see that if $x \in S$, then $\{x\}^u \subseteq S$, and so we always have $1 \in S$. Hence, by Proposition 2.3, one can conclude that $\Gamma_S(\mathcal{L})$ is connect with diameter less than three.

Proposition 3.1 *For all $s \in S$, we have that* $\deg(s) = |\mathcal{L}| - 1$.

Proof Let s be an arbitrary element in S. Since $s \wedge (s \vee x) = s$ for all $x \in \mathcal{L}$, we have that $s \wedge (s \vee x) \in S$. Therefore $\deg(s) = |\mathcal{L}| - 1$ for all $s \in S$. □

In the following proposition, we present a lower bound for the clique number of $\Gamma_S(\mathcal{L})$.

Proposition 3.2 *In the graph $\Gamma_S(\mathcal{L})$ we have the following inequality.*

$$\omega(\Gamma_S(\mathcal{L})) \geq \max\{|S|, |\mathrm{Coatom}(\mathcal{L})|\} + 1$$

Proof Since $m \vee m' = 1$ for all m and m' in $\mathrm{Coatom}(\mathcal{L})$, the set $\mathrm{Coatom}(\mathcal{L}) \cup \{1\}$ forms a clique in $\Gamma_S(\mathcal{L})$. Also, by Proposition 3.1, $S \cup \{0\}$ is a clique in $\Gamma_S(\mathcal{L})$. This implies that $\omega(\Gamma_S(\mathcal{L})) \geq \max\{|S|, |\mathrm{Coatom}(\mathcal{L})|\} + 1$. □

Lemma 3.3 *Assume that $0 \in S$. Then $\Gamma_S(\mathcal{L})$ is complete.*

Proof Since $0 \in S$, we have that $\{0\}^u \subseteq S$. Hence $S = \mathcal{L}$. Now, by Proposition 2.4, the graph $\Gamma_S(\mathcal{L})$ is complete. □

From now on, we assume that $0 \notin S$. So, by Lemma 2.1, we have $|S \cap \mathrm{Atom}(\mathcal{L})| \leq 1$.

Proposition 3.4 *If $S \cap \mathrm{Atom}(\mathcal{L}) = \{a\}$, then $S = \{a\}^u$.*

Proof Clearly $\{a\}^u \subseteq S$. Now, assume in contrary that there exists $b \in S \setminus \{a\}^u$. Since S is a multiplicatively closed subset of $\mathcal{L}$, we have $0 = a \wedge b \in S$, which is impossible. Thus $S = \{a\}^u$. □

Proposition 3.5 *Suppose that $|\mathcal{L}| \geq 3$.*

(i) If $|S| \geq 2$, then every vertex of the graph $\Gamma_S(\mathcal{L})$ lies in a cycle of length 3, and so $\mathrm{gr}(\Gamma_S(\mathcal{L})) = 3$.
(ii) If $|S| = 1$, then $\mathrm{gr}(\Gamma_S(\mathcal{L})) \in \{3, \infty\}$.

Proof (i) Since $|S| \geq 2$, we can choose $s \neq 1$ in S. Now, let x be an arbitrary element in $\mathcal{L}$. Then we have the cycle $1 - x - s - 1$, and so each vertex of the graph $\Gamma_S(\mathcal{L})$ lies in a cycle of length 3 and $\mathrm{gr}(\Gamma_S(\mathcal{L})) = 3$.

(ii) Since $|S| = 1$, we have that $S = \{1\}$. The result now follows from Proposition 2.6. □

Proposition 3.6 *If $|S| = 1$, then* $\deg(x) = 1$, *for all $x \in J(\mathcal{L})$.*

Proof Since $|S| = 1$, we have that $S = \{1\}$. If $|\text{Coatom}(\mathcal{L})| = 1$, then, by Proposition 2.5, $\Gamma_S(\mathcal{L})$ is a star graph with center 1. Since $1 \notin J(\mathcal{L})$, every vertex in $\mathrm{J}(\mathcal{L})$ has degree one. Otherwise, $|\text{Coatom}(\mathcal{L})| \geq 2$. Let x be an arbitrary vertex in $J(\mathcal{L})$. Clearly 1 is adjacent to every vertex in $\Gamma_S(\mathcal{L})$ which implies that $\deg(x) \geq 1$. Assume in contrary that $\deg(x) \geq 2$. So there is $y \neq 1$ in $\mathcal{L}$ such that x and y are adjacent, and so $x \vee y = 1$. If $y \in \text{Coatom}(\mathcal{L})$, then $x \vee y = y$, which is impossible. Hence $y \notin \text{Coatom}(\mathcal{L})$, but there is $m \in \text{Coatom}(\mathcal{L})$ such that $y \in \{m\}^\ell$. Since $x \in \bigcap_{m \in \text{Coatom}(\mathcal{L})} \{m\}^\ell$, we have that $x \vee y \leq m$, which is impossible. So $\deg(x) = 1$, for all $x \in J(\mathcal{L})$. □

Proposition 3.7 *If* $|S| = 1$ *and* $|\text{Coatom}(\mathcal{L})| \geq 2$, *then every vertex in graph* $\Gamma_S(\mathcal{L} \setminus \mathrm{J}(\mathcal{L}))$ *lies in a cycle of length 3.*

Proof Since $|S| = 1$, we have that $S = \{1\}$. Let y be an arbitrary element in $\mathcal{L} \setminus J(\mathcal{L})$. We need to show that y lies in a cycle of length 3 in $\Gamma_S(\mathcal{L} \setminus J(\mathcal{L}))$. If $y \in \text{Coatom}(\mathcal{L})$, since $|\text{Coatom}(\mathcal{L})| \geq 2$, there exists m with $m \neq y$ in $\text{Coatom}(\mathcal{L})$. Thus, we can consider the cycle $y - m - 1 - y$ in $\Gamma_S(\mathcal{L} \setminus J(\mathcal{L}))$. Otherwise $y \notin \text{Coatom}(\mathcal{L})$. Since $y \notin J(\mathcal{L})$, there exists $m \in \text{Coatom}(\mathcal{L})$ such that $y \notin \{m\}^\ell$. Therefore $y \vee m = 1$, which implies that y and m are adjacent in $\Gamma_S(\mathcal{L} \setminus J(\mathcal{L}))$. So we can consider the cycle $y - m - 1 - y$ and the result follows. □

4 Planarity of $\Gamma_S(\mathcal{L})$ when S is a saturated subset of $\mathcal{L}$

In this section, we characterize all planar graphs $\Gamma_S(\mathcal{L})$, where S is a saturated subset of $\mathcal{L}$. Recall that a planar graph is a graph that can be embedded on the plane, that is, it can be drawn on the plane in such a way that it's edges intersect only at their endpoints. Kuratowski provided a nice characterization of planar graphs, which now is known as Kuratowski's Theorem:

A finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proposition 4.1 *If* $0 \in S$, *then* $\Gamma_S(\mathcal{L})$ *is planar if and only if* $|\mathcal{L}| \leq 4$.

Proof If $0 \in S$, then $S = \mathcal{L}$. Hence $\Gamma_S(\mathcal{L})$ is a complete graph, and so the result follows from Kuratowski's Theorem. □

Proposition 3.2 in conjunction with Kuratowski's Theorem implies the following lemma.

Lemma 4.2 *If* $|\text{Coatom}(\mathcal{L})| \geq 4$ *or* $|S| \geq 4$, *then* $\Gamma_S(\mathcal{L})$ *is not planar.*

In view of Lemma 4.2, we investigate to study the planarity of $\Gamma_S(\mathcal{L})$, in the case that $|S| \leqslant 3$ and $|\text{Coatom}(\mathcal{L})| \leqslant 3$. We start with the following proposition.

Proposition 4.3 *Suppose that* $|S| = 3$. *Then* $\Gamma_S(\mathcal{L})$ *is planar if and only if* $|\mathcal{L}| \leq 5$.

Proof Suppose that $\Gamma_S(\mathcal{L})$ is planar. Assume in contrary that $|\mathcal{L}| \geq 6$. Now, put $V_1 := S$ and $V_2 := \{x_1, x_2, x_3\} \subseteq \mathcal{L} \setminus S$. Clearly one can find a copy of $K_{3,3}$ in $\Gamma_S(\mathcal{L})$.

Therefore, by Kuratowski's Theorem, $\Gamma_S(\mathcal{L})$ is not planar, which is impossible. Hence $|\mathcal{L}| \leq 5$.

Conversely, assume that $|\mathcal{L}| \leq 5$. It is clear that if $|\mathcal{L}| \leq 4$, then $\Gamma_S(\mathcal{L})$ is planar. Also, if $|\mathcal{L}| = 5$, then, in view of Proposition 2.4, $\Gamma_S(\mathcal{L})$ is not K_5. So $\Gamma_S(\mathcal{L})$ is planar. □

Note that, if $|S| = 2$, then $S = \{1, m\}$ for some $m \in \mathrm{Coatom}(\mathcal{L})$. Also, if $|S| = 1$, then $S = \{1\}$.

For the rest of the paper, we need the following definition.

Definition 4.4 *Let x be an arbitrary element in $\mathcal{L}$. We define the lower neighbors of x as the set $B_\ell(x) := \{y \in \mathcal{L};\ x\ covers\ y\}$. Also, for every subsets A and B of $\mathcal{L}$, we put $L_A^B := \{B\}^\ell \setminus \{A\}^\ell$. Moreover, for $x \in \mathcal{L}$, we denote the sets $L_A^{\{x\}}$ and $L_{\{x\}}^B$, by L_A^x and L_x^B, respectively.*

Lemma 4.5 *Assume that $S = \{1, m\}$, where $m \in \mathrm{Coatom}(\mathcal{L})$. If $|B_\ell(m)| \geq 3$, then $\Gamma_S(\mathcal{L})$ is not planar.*

Proof Since $|B_\ell(m)| \geq 3$, we can choose the subset $A = \{x_1, x_2, x_3\}$ of the set $B_\ell(m)$. It is clear that $x_i \vee x_j = m$ for all $1 \leq i \neq j \leq 3$. So, the induced subgraph of $\Gamma_S(\mathcal{L})$ on $A \cup S$ is isomorphic to K_5. Thus, by Kuratowski's Theorem, $\Gamma_S(\mathcal{L})$ is not planar. □

Proposition 4.6 *Let $S = \{1, m\}$, where $m \in \mathrm{Coatom}(\mathcal{L})$. If $\Gamma_S(\mathcal{L})$ is planar, then one of the following statements holds:*

(i) $|B_\ell(m)| = 1$,
(ii) $|B_\ell(m)| = 2$, *and* $|L_x^y| \leq 2$, *for all* $x, y \in B_\ell(m)$, *and if* $|L_x^y| = 2$, *then* $|L_y^x| = 1$, *and if* $|L_y^x| = 2$, *then* $|L_x^y| = 1$.

Proof Since $\Gamma_S(\mathcal{L})$ is planar, by Lemma 4.5, $|B_\ell(m)| \leq 2$. Suppose that $|B_\ell(m)| \neq 1$. Thus, we have that $|B_\ell(m)| = 2$. Set $B_\ell(m) := \{x, y\}$. If $|L_y^x| \geq 3$ or $|L_x^y| \geq 3$, then it is easy to see that $\Gamma_S(\mathcal{L})$ has a subgraph which is isomorphic to $K_{3,3}$, and so $\Gamma_S(\mathcal{L})$ is not planar, which is impossible. Hence $|L_y^x| \leq 2$ and $|L_x^y| \leq 2$.

Now, suppose that $|L_x^y| = 2$ and $|L_y^x| = 2$. Set $V_1 := L_x^y \cup \{m\}$ and $V_2 := L_y^x \cup \{1\}$. It is easy to see that $\Gamma_S(\mathcal{L})$ has a subgraph isomorphic to $K_{3,3}$ with parts V_1 and V_2, which is impossible. So if $|L_x^y| = 2$, then $|L_y^x| = 1$. Also, if $|L_y^x| = 2$, then $|L_x^y| = 1$. □

Assume that $S = \{1, m\}$, where $m \in \mathrm{Coatom}(\mathcal{L})$. By Lemma 4.2, we have $|\mathrm{Coatom}(\mathcal{L})| \leq 3$. In the following proposition, we present a necessary and sufficient condition for planarity of $\Gamma_S(\mathcal{L})$ whenever $|\mathrm{Coatom}(\mathcal{L})| = 1$.

Theorem 4.7 *Let $S = \{1, m\}$ and $\mathrm{Coatom}(\mathcal{L}) = \{m\}$. Then the graph $\Gamma_S(\mathcal{L})$ is planar if and only if one of the conditions (i) or (ii) in Proposition 4.6 holds.*

Proof Let $\Gamma_S(\mathcal{L})$ is planar. Then, by Proposition 4.6, we are done.

Conversly, we show that if one of the conditions (i) or (ii) in Proposition 4.6 occurs, then $\Gamma_S(\mathcal{L})$ is planar. One can easily check that, if $|B_\ell(m)| = 1$, then $\Gamma_S(\mathcal{L})$ is isomorphic to $K_{1,1,|\mathcal{L}|-2}$, and so $\Gamma_S(\mathcal{L})$ is planar. Also, if condition (ii) in Proposition 4.6 occurs, then it is not hard to see that $\Gamma_S(\mathcal{L})$ is planar. □

In the next theorem, we provide a necessary and sufficient condition for planarity of $\Gamma_S(\mathcal{L})$ whenever $|\mathrm{Coatom}(\mathcal{L})| = 2$.

Theorem 4.8 *Suppose that* $S = \{1, m\}$ *and* $\mathrm{Coatom}(\mathcal{L}) = \{m, m'\}$. *Then the graph* $\Gamma_S(\mathcal{L})$ *is planar if and only if* $\mathcal{L}$ *is one of the following lattices:*

(i) $\mathcal{L} = S \cup L^x_{m'} \cup \mathrm{J}(\mathcal{L})$, *where* $B_\ell(m) = \{x\}$ *such that* $x \in \{m'\}^\ell$.

(ii) $\mathcal{L} = S \cup L^{m'}_x \cup L^x_{m'} \cup \mathrm{J}(\mathcal{L})$, *where* $B_\ell(m) = \{x\}$ *such that*

(a) $x \notin \{m'\}^\ell$.

(b) The numbers $|L^{m'}_x|$ *and* $|L^x_{m'}|$ *are less than 3, and if* $|L^{m'}_x| = 2$, *then* $|L^x_{m'}| = 1$, *and if* $|L^x_{m'}| = 2$, *then* $|L^{m'}_x| = 1$.

(iii) $\mathcal{L} = S \cup \{m'\} \cup B_\ell(m) \cup \mathrm{J}(\mathcal{L})$, *where* $|B_\ell(m)| = 2$ *and* $|B_\ell(m) \setminus \{m'\}^\ell| = 1$.

Proof Assume that $\Gamma_S(\mathcal{L})$ is a planar graph. Then, by Lemma 4.6, $|B_\ell(m)\setminus\{m'\}^\ell| \leq 2$. Now, suppose that $|B_\ell(m) \setminus \{m'\}^\ell| = 2$, and set $B_\ell(m) \setminus \{m'\}^\ell := \{x, y\}$. Since the induced subgraph of $\Gamma_S(\mathcal{L})$ on $\{1, m, m', x, y\}$ is isomorphic to K_5, by Kuratowski's Theorem, $\Gamma_S(\mathcal{L})$ is not planar, which is impossible. Thus $|B_\ell(m) \setminus \{m'\}^\ell| \leq 1$. Also, by Proposition 4.6, one of the conditions (i) or (ii) in Proposition 4.6 must holds. So we have the following cases:

Case 1. Suppose that the condition (i) in Proposition 4.6 holds and put $B_\ell(m) := \{x\}$. At first note that $\mathcal{L} = S \cup L^{m'}_x \cup L^x_{m'} \cup \mathrm{J}(\mathcal{L})$. Now, assume that $|B_\ell(m) \setminus \{m'\}^\ell| = 1$. Thus $B_\ell(m) \setminus \{m'\}^\ell = \{x\}$. If $|L^{m'}_x| \geq 3$, then it is easy to see that the graph $\Gamma_S(\mathcal{L})$ has a copy of $K_{3,3}$, which is impossible. Therefore $|L^{m'}_m| \leq 2$. Similarly $|L^x_{m'}| \leq 2$. Now, if $|L^{m'}_x| = |L^x_{m'}| = 2$, then consider the sets $V_1 := L^{m'}_x \cup \{1\}$ and $V_2 := L^x_{m'} \cup \{m\}$, to deduce that $\Gamma_S(\mathcal{L})$ has a copy of $K_{3,3}$, which is impossible. Therefore, if $|L^{m'}_x| = 2$, then $|L^x_{m'}| = 1$. Similarly, if $|L^x_{m'}| = 2$, then $|L^{m'}_x| = 1$. So $\mathcal{L} = S \cup L^{m'}_x \cup L^x_{m'} \cup \mathrm{J}(\mathcal{L})$, where $B_\ell(m) = \{x\}$ and $|L^{m'}_x|$, and $|L^x_{m'}|$ are less than 3, and if $|L^{m'}_x| = 2$, then $|L^x_{m'}| = 1$, and also, if $|L^{m'}_x| = 2$, then $|L^x_{m'}| = 1$.

Now, let $|B_\ell(m) \setminus \{m'\}^\ell| = 0$. Since $|B_\ell(m) \setminus \{m'\}^\ell| = 0$, we have that $L^x_{m'} = \emptyset$. So $\mathcal{L} = S \cup L^{m'}_x \cup \mathrm{J}(\mathcal{L})$.

Case 2. Suppose that the condition (ii) in Proposition 4.6 holds. Set $B_\ell(m) := \{x, y\}$. Note that in this case, we have that

$$\mathcal{L} = S \cup L^{m'}_{\{x,y\}} \cup L^x_{\{m',y\}} \cup L^y_{\{m',x\}} \cup L^{\{m',x\}}_y \cup L^{\{m',y\}}_x \cup L^{\{x,y\}}_{m'} \cup \mathrm{J}(\mathcal{L}).$$

Since $\mathcal{L}$ is a lattice, we have that $|B_\ell(m)\setminus\{m'\}^\ell| = 1$. Without loss of generality, we may assume that $B_\ell(m)\setminus\{m'\}^\ell = \{y\}$. If $|L^{m'}_{\{x,y\}}| \geq 2$, then, by setting $V_1 := L^{m'}_{\{x,y\}} \cup \{y\}$ and $V_2 := \{1, m, x\}$, one can find a copy of $K_{3,3}$ in the graph $\Gamma_S(\mathcal{L})$, which is impossible. Hence $L^{m'}_{\{x,y\}} = \{m'\}$. Since $x \in \{m'\}^\ell$, we have that $L^x_{\{y,m'\}} = \emptyset$. Also, if $|L^y_{\{x,m'\}}| \geq 2$, then one can easily to find a copy of $K_{3,3}$, which is impossible. So $L^y_{\{x,m'\}} = \{y\}$. On the other hand, since $x \in \{m'\}^\ell$, $L^{\{x,y\}}_{m'} = \emptyset$. Thus $\mathcal{L} = S \cup \{m'\} \cup B_\ell(m) \cup \mathrm{J}(\mathcal{L})$.

The converse statment is clear. $\square$

In the following, we investigate the planarity of the graph $\Gamma_S(\mathcal{L})$ whenever $|S| = 2$ and $\mathrm{Coatom}(\mathcal{L}) = \{m, m', m''\}$.

Proposition 4.9 *Suppose that* $S = \{1, m\}$, $\mathrm{Coatom}(\mathcal{L}) = \{m, m', m''\}$ *and that the graph* $\Gamma_S(\mathcal{L})$ *is planar. Then the following statements hold.*

(i) $L^m_{\{m',m''\}} = \{m\}$,
(ii) $|L^{m'}_{\{m,m''\}}| \leq 2$ *and* $|L^{m''}_{\{m,m'\}}| \leq 2$, *and if* $|L^{m'}_{\{m,m''\}}| = 2$ *then* $|L^{m''}_{\{m,m'\}}| = 1$, *and if* $|L^{m''}_{\{m,m'\}}| = 2$, *then* $|L^{m'}_{\{m,m''\}}| = 1$

Proof (i) Suppose that $\Gamma_S(\mathcal{L})$ is planar. Assume in contrary that $|L^m_{\{m',m''\}}| \geq 2$. Note that, for every element $x \in L^m_{\{m',m''\}}$, we have $x \vee m' = 1$ and $x \vee m'' = 1$. So the graph $\Gamma_S(\mathcal{L})$ has a copy of K_5, which is impossible. Thus $L^m_{\{m',m''\}} = \{m\}$.

(ii) Assume in contrary that $|L^{m'}_{\{m,m''\}}| \geq 3$ or $|L^{m''}_{\{m,m'\}}| \geq 3$. One can easily see that the graph $\Gamma_S(\mathcal{L})$ has a subgraph isomorphic to $K_{3,3}$, and so $\Gamma_S(\mathcal{L})$ is not planar, which is impossible. Therefore $|L^{m'}_{\{m,m''\}}| \leq 2$ and $|L^{m''}_{\{m,m'\}}| \leq 2$. Now, if $|L^{m'}_{\{m,m''\}}| = 2$ and $|L^{m''}_{\{m,m'\}}| = 2$, then, by setting $V_1 := L^{m''}_{\{m,m'\}} \cup \{1\}$ and $V_2 := L^{m'}_{\{m,m''\}} \cup \{m\}$, we can find a copy of $K_{3,3}$ in the graph $\Gamma_S(\mathcal{L})$, which is impossible. Hence if $|L^{m'}_{\{m,m''\}}| = 2$, then $|L^{m''}_{\{m,m'\}}| = 1$. Similarly if $|L^{m''}_{\{m,m'\}}| = 2$, then $|L^{m'}_{\{m,m''\}}| = 1$. □

Theorem 4.10 *Let* $S = \{1, m\}$ *and* $\mathrm{Coatom}(\mathcal{L}) = \{m, m', m''\}$. *Then the graph* $\Gamma_S(\mathcal{L})$ *is planar if and only if* $\mathcal{L}$ *is one of the following lattices:*

(i) $\mathcal{L} = \{1, m, m', m'', x\} \cup L_x^{\{m',m''\}} \cup \mathrm{J}(\mathcal{L})$, *where* $B_\ell(m) = \{x\}$ *and* $x \in \{m'\}^\ell \backslash \{m''\}^\ell$ *or* $x \in \{m''\}^\ell \backslash \{m'\}^\ell$.
(ii) $\mathcal{L} = \{1, m, x\} \cup L^{m'}_{\{x,m''\}} \cup L^{m'}_{\{x,m''\}} \cup L_x^{\{m',m''\}} \cup \mathrm{J}(\mathcal{L})$, *where* $B_\ell(m) = \{x\}$ *and*
  (a) $x \in \{m'\}^\ell \cap \{m''\}^\ell$,
  (b) $|L^{m'}_{\{x,m''\}}| \leq 2$ *and* $|L^{m'}_{\{x,m''\}}| \leq 2$, *and if* $|L^{m'}_{\{x,m''\}}| = 2$, *then* $|L^{m''}_{\{x,m'\}}| = 1$, *and if* $|L^{m''}_{\{x,m'\}}| = 2$, *then* $|L^{m'}_{\{x,m''\}}| = 1$

Proof Suppose that $\Gamma_S(\mathcal{L})$ is planar. Firstly, by Proposition 4.9 , we have $L^m_{\{m',m''\}} = \{m\}$ which implies that $|B_\ell(m) \setminus \{m', m''\}^\ell| = 0$. Now, by Proposition 4.6, we have the following cases:

Case 1. Suppose that the condition (i) of Proposition 4.6 holds. Then set $B_\ell(m) := \{x\}$. Since $|B_\ell(m)\backslash\{m', m''\}^\ell| = 0$, we have the following subcases:

(a) $B_\ell(m)\backslash\{m'\}^\ell = \{x\}$ and $B_\ell(m)\backslash\{m''\}^\ell = \emptyset$. In this case, if $|L^{m'}_{\{x,m''\}}| \geq 2$, then one can easily find a copy of $K_{3,3}$, which is impossible. So $|L^{m'}_{\{x,m''\}}| = 1$. Similarly $|L^{m''}_{\{x,m'\}}| = 1$. Also $L^x_{\{m',m''\}}$, $L^{\{x,m'\}}_{m''}$ and $L^{\{x,m''\}}_{m'}$ are empty sets, because $x \in \{m'\}^\ell$. So

$$\mathcal{L} = \{1, m, m', m'', x\} \cup L_x^{\{m',m''\}} \cup \mathrm{J}(\mathcal{L}).$$

(b) $B_\ell(m)\backslash\{m''\}^\ell = \{x\}$ and $B_\ell(m)\backslash\{m'\}^\ell = \emptyset$. It is similar to (a).
(c) $B_\ell(m)\backslash\{m''\}^\ell = \emptyset$ and $B_\ell(m)\backslash\{m'\}^\ell = \emptyset$. In this case, by Proposition 4.9, we have that $|L^{m'}_{\{x,m''\}}| \leq 2$ and $|L^{m'}_{\{x,m''\}}| \leq 2$, and if $|L^{m'}_{\{x,m''\}}| = 2$, then $|L^{m''}_{\{x,m'\}}| = 1$

and vis versa. Also, since $x \in \{m'\}^\ell$ and $x \in \{m''\}^\ell$, the sets $L^x_{\{m',m''\}}$, $L^{\{x,m'\}}_{m''}$ and $L^{\{x,m'\}}_{m'}$ are empty. Thus $\mathcal{L} = \{1, m, x\} \cup L^{m'}_{\{x,m''\}} \cup L^{m'}_{\{x,m''\}} \cup L^{\{m',m''\}}_x \cup \mathrm{J}(\mathcal{L})$, where $|L^{m'}_{\{x,m''\}}| \leq 2$ and $|L^{m'}_{\{x,m''\}}| \leq 2$, and if $|L^{m'}_{\{x,m''\}}| = 2$ and $|L^{m''}_{\{x,m'\}}| = 1$ and if $|L^{m''}_{\{x,m'\}}| = 2$ and $|L^{m'}_{\{x,m''\}}| = 1$

Case 2. Suppose that the condition (ii) of Proposition 4.6 holds. Set $B_\ell(m) := \{x, y\}$. Since $\mathcal{L}$ is a lattice and $|B_\ell(m) \setminus \{m', m''\}^\ell| = 0$, one can conclude that $|B_\ell(m) \setminus \{m'\}^\ell| = 1$ and $|B_\ell(m) \setminus \{m''\}^\ell| = 1$. So, without loss the generality, we may assume that $x \in L^{m'}_{m''}$ and $y \in L^{m''}_{m'}$. Now if we set $V_1 := \{1, m', x\}$ and $V_1 := \{m, m'', y\}$, then one can find a subgraph isomorphic to $K_{3,3}$ in $\Gamma_S(\mathcal{L})$ which is impossile. So this case never happens.

The converse statment is clear. □

Now, the only remaining case is that $|S| = 1$. The following lemmas are useful.

Lemma 4.11 *Assume that* $|S| = 1$. *Then the graph* $\Gamma_S(\mathcal{L})$ *is planar if and only if* $\Gamma_S(\mathcal{L} \setminus J(\mathcal{L}))$ *is planar.*

Proof By Proposition 3.6 we have that $\deg(x) = 1$, for all $x \in J(\mathcal{L})$. So, the result holds. □

Lemma 4.12 *Let* $|S| = 1$ *and* $|\mathrm{Coatom}(\mathcal{L})| \geq 2$. *If* $|L^m_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \geq 3$ *and* $|L^{m'}_m| \geq 2$, *for some distinct* $m, m' \in \mathrm{Coatom}(\mathcal{L})$, *then* $\Gamma_S(\mathcal{L})$ *is not planar.*

Proof Since $|L^m_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \geq 3$, we set $V_1 := \{m, x_1, x_2\} \subseteq L^m_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}$. Also, since $|L^{m'}_m| \geq 2$, we set $V_2 := \{1, m', y_1\}$ where $y_1 \in L^{m'}_m$. It is easy to see that the subgraph of $\Gamma_S(\mathcal{L})$ on $V_1 \cup V_2$ is isomorphic to $K_{3,3}$. Thus, by Kuratowski's Theorem, $\Gamma_S(\mathcal{L})$ is not planar. □

In the next theorem, we provide a necessary and sufficient condition for planarity of $\Gamma_S(\mathcal{L})$ when $|S| = 1$.

Theorem 4.13 *Suppose that* $|S| = 1$. *Then the graph* $\Gamma_S(\mathcal{L})$ *is planar if and only if one of the following statements holds:*

(i) $|\mathrm{Coatom}(\mathcal{L})| = 1$,
(ii) $|\mathrm{Coatom}(\mathcal{L})| = 2$ *and if there is an element* $m \in \mathrm{Coatom}(\mathcal{L})$ *such that* $|L^m_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \geq 3$, *then* $|L_m^{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| = 1$
(iii) $|\mathrm{Coatom}(\mathcal{L})| = 3$, *and*
  (a) For all $m \in \mathrm{Coatom}(\mathcal{L})$, $|L^m_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \leq 2$ *and there is at most one element* m *in* $\mathrm{Coatom}(\mathcal{L})$ *such that* $|L^m_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| = 2$.
  (b) If there is $m \in \mathrm{Coatom}(\mathcal{L})$ *such that* $|L^m_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| = 2$, *then* $|L_m^{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| = 0$.

Proof Assume that the graph $\Gamma_S(\mathcal{L})$ is planar. Then, by Proposition 3.2, $|\mathrm{Coatom}(\mathcal{L})| \leq 3$. Suppose that $|\mathrm{Coatom}(\mathcal{L})| \neq 1$. So, we have the following cases:

Case 1. Suppose that $|\mathrm{Coatom}(\mathcal{L})| = 2$ and put $\mathrm{Coatom}(\mathcal{L}) = \{m, m'\}$. If there is one element $c \in \mathrm{Coatom}(\mathcal{L})$ such that $|L^{c}_{\mathrm{Coatom}(\mathcal{L})\setminus\{c\}}| \geq 3$, then we show that $|L_c^{\mathrm{Coatom}(\mathcal{L})\setminus\{c\}}| = 1$. Without loss the generality, we may assume that $c = m$. Assume in contrary that $|L_m^{m'}| \geq 2$. Then, by setting $V_1 := \{1, m', x\}$ and $V_2 := \{m, y_1, y_2\}$, where $x \in L_m^{m'}$ and $y_1, y_2 \in L^{m}_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}$, one can find a copy of $K_{3,3}$ in $\Gamma_S(\mathcal{L})$, which is impossible. Therefore if there is one element $m \in \mathrm{Coatom}(\mathcal{L})$ such that $|L^{m}_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \geq 3$, then $L_m^{m'} = \{m'\}$.

Case 2. Suppose that $|\mathrm{Coatom}(\mathcal{L})| = 3$ and put $\mathrm{Coatom}(\mathcal{L}) = \{m, m', m''\}$. It is easy to see that if $|L^{m}_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \geq 3$, then the graph $\Gamma_S(\mathcal{L})$ is not planar, which is impossible. So $|L^{m}_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \leq 2$ for all $m \in \mathrm{Coatom}(\mathcal{L})$. Now, Assume in contrary that $|L^{c}_{\mathrm{Coatom}(\mathcal{L})\setminus\{c\}}| = 2$ and $|L^{c'}_{\mathrm{Coatom}(\mathcal{L})\setminus\{c'\}}| = 2$ for some c and c' in $\mathrm{Coatom}(\mathcal{L})$. Without loss of generality, we may assume that $c = m$ and $c' = m'$. Put $V_1 := \{1, m, x\}$ and $V_2 := \{m', m'', y\}$, where $x \in L^{m}_{\{m',m''\}}$ and $y \in L^{m'}_{\{m,m''\}}$. Then one can find a subgraph of $\Gamma_S(\mathcal{L})$ which is isomorphic to $K_{3,3}$, which is impossible. So, there is at most one element c in $\mathrm{Coatom}(\mathcal{L})$ such that $|L^{c}_{\mathrm{Coatom}(\mathcal{L})\setminus\{c\}}| = 2$. Now, assume that $|L^{m}_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| = 2$, and assume in contrary that $|L_m^{\{m',m''\}}| \geq 1$. Then, by setting $V_1 := \{1\} \cup L^{m}_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}$ and $V_2 := \{m', m'', y\}$ where $y \in L_m^{\{m',m''\}}$, we can find a subgraph in the graph $\Gamma_S(\mathcal{L})$ which is isomorphic to $K_{3,3}$, which is impossible. So, $|L_m^{\{m',m''\}}| = 0$.

Conversely, if $|\mathrm{Coatom}(\mathcal{L})| = 1$, then, by part (ii) of Proposition 2.5, $\Gamma_S(\mathcal{L})$ is a star graph, and so it is planar.

Now, assume that $|\mathrm{Coatom}(\mathcal{L})| = 2$ and that there is one element $m \in \mathrm{Coatom}(\mathcal{L})$ such that $|L^{m}_{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| \geq 3$. Then $|L_m^{\mathrm{Coatom}(\mathcal{L})\setminus\{m\}}| = 1$. It is not hard to see that $\Gamma_S(\mathcal{L} \setminus \mathcal{J}(\mathcal{L}))$ is a complete 3-partite graph such that at least two parts of it has exactly one element. Therefore $\Gamma_S(\mathcal{L} \setminus \mathcal{J}(\mathcal{L}))$ is planar. Now, by Proposition 4.11, $\Gamma_S(\mathcal{L})$ is planar.

Now, suppose that $|\mathrm{Coatom}(\mathcal{L})| = 3$ and put $\mathrm{Coatom}(\mathcal{L}) = \{m, m', m''\}$. Also, assume that the conditions (a) and (b) hold. It is easy to see that the subgraph $\Gamma_S(\mathcal{L} \setminus \mathcal{J}(\mathcal{L}))$ on $\{1\} \cup L^{m}_{\{m',m''\}} \cup L^{m'}_{\{m,m''\}} \cup L^{m''}_{\{m,m'\}}$ is a complete 4-partite graph such that at least three parts of it has exactly one element. Now, by condition (b), it is easy to see that $\Gamma_S(\mathcal{L} \setminus \mathcal{J}(\mathcal{L}))$ is planar. Hence, by Proposition 4.11, $\Gamma_S(\mathcal{L})$ is planar. □

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